ARL-TR-10368 ● JUNE 2026

Lagrangian Interpolation of Nonuniform Signals from Level-Crossing Measurements

by Vinod K. Mishra

NOTICES

Disclaimers

ARL-TR-10368 ● JUNE 2026

Lagrangian Interpolation of Nonuniform Signals from Level-Crossing Measurements

Vinod K. Mishra
DEVCOM Army Research Laboratory

REPORT DOCUMENTATION PAGE

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED	
		START DATE	END DATE
June 2026	Technical Report	2/15/2026	4/15/2026

4. TITLE AND SUBTITLE
Lagrangian Interpolation of Nonuniform Signals from Level-Crossing Measurements

5a. CONTRACT NUMBER	5b. GRANT NUMBER	5c. PROGRAM ELEMENT NUMBER
5d. PROJECT NUMBER	**5e. TASK NUMBER**	**5f. WORK UNIT NUMBER**

6. AUTHOR(S)
Vinod K. Mishra

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
DEVCOM Army Research Laboratory ATTN: FCDD-RLA-NC Aberdeen Proving Ground, MD 21005	ARL-TR-10368

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
DISTRIBUTION STATEMENT A. Approved for public release: distribution unlimited.

13. SUPPLEMENTARY NOTES
ORCID: Vinod K. Mishra, 0000-0001-9432-9082

14. ABSTRACT
The signals, for example, communication, radar, electrocardiogram, etc., are obtained using discrete time methods. Recently, level-crossing-based continuous time-digital signal processing (CT-DSP) methods provided many advantages like lower signal-to-noise ratio, no aliasing, lower processing need, and so on. There are not many effective methods for obtaining continuous signals from discrete nonuniform time measurements. This work applies the Lagrangian interpolation method to this problem and shows that it leads to a good reconstruction of CT-DSP signals.

15. SUBJECT TERMS
signal processing, continuous time, CT, digital signal processing, DSP, Lagrangian interpolation, nonuniform sampling, Network, Cyber, and Computational Sciences

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES
a. REPORT	b. ABSTRACT	C. THIS PAGE		
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UU	19

19a. NAME OF RESPONSIBLE PERSON	19b. PHONE NUMBER *(Include area code)*
Vinod K. Mishra	(520) 691-5761

STANDARD FORM 298 (REV. 5/2020)
Prescribed by ANSI Std. Z39.18

Contents

List of Figures

List of Tables

1. Introduction

Signals in the tactical arena are of many kinds and vary in their characteristics across a wide range, for example, communication, sensing, radar signals, etc. Signal processing approaches applicable to these signals use either discrete- or continuous-time (CT) sampling. The last approach is based on continuous time,[1–4] which is relatively new. It has shown distinct advantages in digital signal processing (DSP), analysis of physiological signals such as electrocardiograms,[5,6] and reconstruction of level-crossing (LC) signals.[7,8] The last problem of reconstruction has been investigated extensively using numerical techniques.[9–13]

In a previous study, I gave an analytical approach to calculate the LC of these signals.[14] In this work, I present an alternative and computationally more accurate method.

2. Convolved Asinc for Uniform Sampling

The discrete time DSP (DT-DSP) is the standard approach to signal processing. It uses ideas of Nyquist rate, sinc (short for cardinal sine) functions, etc. For DT, the uniform signal can be characterized by $s_{DT}(kT)$, with k as an integer, and T as the sample spacing.

The following relation connects a DT signal $\hat{g}[n]$ to a corresponding uniformly sampled CT signal $s(t)$:

$$s(t) = \sum_{n=0}^{N-1} \hat{g}[n]\, h(t - nT_s) \qquad (1)$$

The analytical h-function depends on the nature and properties of the signal. For the uniform time instants in $h(t_n)$, we have $t_n = t - nT_s$, where T_s is the sampling period. Mathematical considerations lead to a specific function known as aliased cardinal sine, or asinc function. It was successfully used in the recently developed progressive asinc reconstruction (PAR) algorithm[7,8] using periodic windows.

The expressions for the convolved asinc signal and its slope are given below for later comparison with the nonuniform case.

2.1 Convolved Asinc Signal

The asinc functions are used in the reconstruction of the CT signals from DT signals. They are the basis of the digital-to-analog conversion (DAC) of the signals. They are also known as LC-sampled (LCS) signals.

Given the fundamental frequency ω_0, the waveform period is

$$T_p = (2\pi/\omega_0) \tag{2}$$

and sample spacing is

$$T_s = \frac{T_p}{N} \tag{3}$$

such that $T_s < (\pi/\omega_{max})$, and ω_{max} is the maximum frequency of the waveform. The signal is represented as (with $a = \frac{\pi}{T_p}$):

$$s(t) = \sum_{n=0}^{N-1} \hat{g}[n]\, h(t_n) \tag{4}$$

Here, $\hat{g}[n]$ is the signal amplitude at *n*-th instant and

$$h(t) = \frac{1}{N}\sin(Nat)\cot(at) \tag{5}$$

For uniform sampling $t_n = t - nT_s$, and so we have

$$s(t) = \sum_{n=0}^{N-1} \hat{g}[n]\, h(t - nT_s) = \frac{1}{N}\sum_{n=0}^{N-1} \hat{g}[n]\sin(Na(t - nT_s))\cot(a(t - nT_s)) \tag{6}$$

Sometimes, the signal is expressed in a different but equivalent way. Let $\alpha = \frac{\pi}{T_s}$, then for even *N*:

$$h(t) = \left(\frac{1}{N}\right)\frac{cos\,(\pi t/T_p)}{sin\,(\pi t/T_p)} sin\,(\pi t/T_s) = \left(\frac{1}{N}\right)\frac{cos\,(\alpha t/N)}{sin\,(\alpha t/N)} sin\,(\alpha t) \tag{7}$$

So, the signal becomes

$$s(t) = \sum_{n=0}^{N-1} \hat{g}[n]\left(\frac{1}{N}\right)\frac{\cos((\alpha t - n\pi)/N)}{sin\,((\alpha t - n\pi)/N)} si\,n(\alpha t - n\pi) \tag{8}$$

Figure 1 shows the periodic nature of this signal.

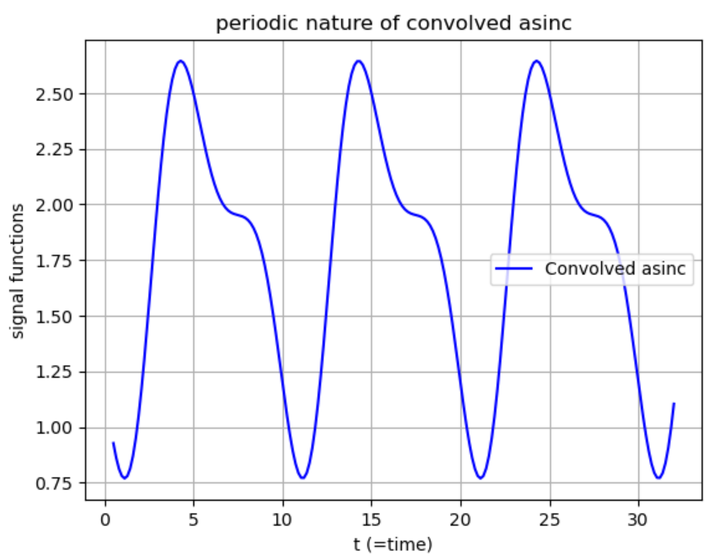

Figure 1. Illustration of the periodic nature of the convolved asinc function with the periodicity in time = 10.002.

2.2 Slope of the Convolved Asinc Signal

Equation 6 is used to calculate the slope of the signal:

$$\frac{ds(t)}{dt} = \sum_{n=0}^{N-1} \hat{g}[n] \frac{d}{dt} h(t - nT_s) \tag{9}$$

using

$$\frac{d}{dt} h(t) = a \left[\cos(Nat) \cot(at) - \frac{1}{N} \sin(Nat) \mathrm{cosec}^2(at) \right] \tag{10}$$

The final expression for the slope becomes

$$\frac{ds(t)}{dt} = \frac{a}{N} \sum_{n=0}^{N-1} \hat{g}[n] \left[N \cos(Na(t - nT_s)) \cot(a(t - nT_s)) - \sin(Na(t - nT_s)) \mathrm{cosec}^2(a(t - nT_s)) \right] \tag{11}$$

At $t = mT_s$, we get

$$\frac{d}{dt} h(mT_s) = a(-1)^m \cot\left(\frac{m\pi}{N}\right), m = 1, \ldots, N \tag{12}$$

$$\frac{d}{dt} h(mT_s) = 0 \; for \; m = 0 \tag{13}$$

This expression facilitates PAR reconstruction.

3. Lagrangian Polynomials Nonuniform Sampling

The CT-DSP signals are sampled at the exact times when the signal amplitude reaches a preset value. This event is called "Level Crossing." The corresponding time values are not equally spaced and have a nonuniform time distribution. It is not straightforward to convert these pairs of amplitude–time pairs and reconstruct a continuous signal.

The signal amplitude level for LC at nonuniform time $t_n$is $s_{CT}(t_n) = \hat{g}[n]$. The following relation connects $\hat{g}[n]$ to a corresponding CT signal $s(t)$ for the nonuniform case.

$$s(t) = \sum_{n=0}^{N-1} \hat{g}[n] \, h(t_n) \tag{14}$$

The analytical h-function depends on the nature and properties of the signal. For nonuniform LC instants t_n, there is no standard approach comparable to uniform sampling for determining $h(t_n)$. Methods like sinc,[11] sinh[13] interpolations, and others[10,12] have been tried and suggested with varying degrees of success. The Lagrange polynomials used as interpolation functions are better known among them and have been chosen for the current work.

3.1 Expression for the Reconstructed Signal from Lagrangian Interpolation

The continuous signal obtained from N-point Lagrangian interpolation formula can be expressed as

$$s(t) = \sum_{n=0}^{N-1} \frac{g[n]}{d_n} \frac{L_N(t)}{(t-t_n)} \tag{15}$$

Here , $g[n]$ is the signal amplitude corresponding to the nonuniform times t_n, and

$$\frac{L_N(t)}{(t-t_n)} = (t-t_0)\dots(t-t_{n-1})(t-t_{n+1})\dots(t-t_{N-1}) \tag{16}$$

$$d_n = (t_n-t_0)\dots(t_n-t_{n-1})(t_n-t_{n+1})\dots(t_n-t_{N-1}) \tag{17}$$

Note that for the term containing $g[n]$, numerator lacks $(t-t_n)$, and in the denominator t is replaced by t_n. In the rest of the report, we will use N = 6. Let

$$G_n = \frac{g[n]}{d_n} \tag{18}$$

Then, expanding the above, we get six terms as

$$\begin{aligned} s(t) &= G_0(t-t_1)(t-t_2)(t-t_3)(t-t_4)(t-t_5) + G_1(t-t_0)(t-t_2)(t-t_3)(t-t_4)(t-t_5) \\ &+G_2(t-t_0)(t-t_1)(t-t_3)(t-t_4)(t-t_5) + G_3(t-t_0)(t-t_1)(t-t_2)(t-t_4)(t-t_5) \\ &+G_4(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_5) + G_5(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_4) \end{aligned} \tag{19}$$

Define a general function to help write the above in a condensed form.

$$D_{abcd}(t) = (t-t_a)(t-t_b)(t-t_c)(t-t_d) \tag{20}$$

Then the Lagrangian interpolation signal expression above is rewritten as

$$\begin{aligned} s(t) = {} & [G_0D_{1234}(t) + G_1D_{0234}(t) + G_2D_{0134}(t) + G_3D_{0124}(t) + G_4D_{0123}(t)] \\ & (t-t_5) + G_5D_{0123}(t)(t-t_4) \end{aligned} \tag{21}$$

After expanding the denominators, the above expression takes the following final form.

$$\begin{aligned} s(t) = {} & g[0]\frac{(t-t_1)(t-t_2)(t-t_3)(t-t_4)(t-t_5)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)(t_0-t_4)(t_0-t_5)} + g[1]\frac{(t-t_0)(t-t_2)(t-t_3)(t-t_4)(t-t_5)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)(t_1-t_4)(t_1-t_5)} \\ & + g[2]\frac{(t-t_0)(t-t_1)(t-t_3)(t-t_4)(t-t_5)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)(t_2-t_4)(t_2-t_5)} + g[3]\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_4)(t-t_5)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)(t_3-t_4)(t_3-t_5)} \\ & + g[4]\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_5)}{(t_4-t_0)(t_4-t_1)(t_4-t_2)(t_4-t_3)(t_4-t_5)} + g[5]\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_4)}{(t_5-t_0)(t_5-t_1)(t_5-t_2)(t_5-t_3)(t_5-t_4)} \end{aligned} \tag{22}$$

This quintic polynomial for six chosen LC instances is used for further computations.

3.2 Expression for the Slope of the Lagrangian Interpolation Signal

The slope for the interpolation curve is obtained by a simple differentiation.

$$\frac{ds(t)}{dt} = \sum_{n=0}^{N-1} \frac{g[n]}{d_n} \frac{d}{dt} \left[\frac{L_N(t)}{(t-t_n)} \right] \tag{23}$$

Using the $D_{abcd}(t)$ defined earlier, we get

$$\begin{aligned}\frac{ds(t)}{dt} &= (G_4 + G_5)D_{0123}(t) + (G_3 + G_5)D_{0124}(t) + (G_3 + G_4)D_{0125}(t) + (G_2 + G_5)D_{0134}(t) \\ &+ (G_2 + G_4)D_{0135}(t) + (G_2 + G_3)D_{0145}(t) + (G_1 + G_5)D_{0234}(t) + (G_1 + G_4)D_{0235}(t) \\ &+ (G_1 + G_3)D_{0245}(t) + (G_1 + G_2)D_{0345}(t) + (G_0 + G_5)D_{1234}(t) + (G_0 + G_4)D_{1235}(t) \\ &+ (G_0 + G_3)D_{1245}(t) + (G_0 + G_2)D_{1345}(t) + (G_0 + G_1)D_{2345}(t)\end{aligned} \tag{24}$$

4. Computational Results

The computational results presented here give full comparison between the input asinc curve and its Lagrangian interpolation in two segments.

We have the following input for the convolved asinc:

LC values: $(\hat{g}[0], \hat{g}[1], \hat{g}[2], \hat{g}[3], \hat{g}[4], \hat{g}[5]) = (1.2, 0.9, 2.3, 2.5, 2, 1.9)$

Corresponding time values: (0, 1.667, 3.333, 5, 6.667, 8.333, 10)

There are 12 LC points on this function divided into two groups of 6 LC points each.

4.1 First Six LC Intersection Points

4.1.1 Lagrangian Interpolation Input

Time values are obtained by inverse calculation from the convolved asinc function given above. They are as follows:

(time, LC): (0.5615, 0.9), (2.1164, 1.2), (2.4505, 1.5), (2.76, 1.8), (3.084, 2.1) (3.4822, 2.4)

4.1.2 Function Plots

Figure 2 shows the Lagrangian interpolation results for the first six points. The Lagrangian curve reproduces the original signal quite faithfully.

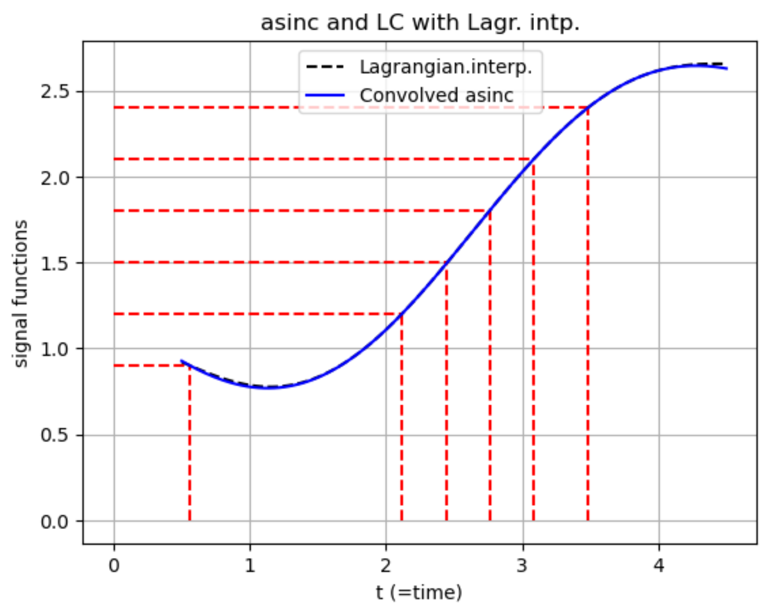

Figure 2. Convolved asinc function and its Lagrangian interpolation for the first six points.

4.1.3 Slope Values

Table 1 shows the original and calculated slope values. They are almost the same except for the first point.

Table 1. Slopes of the convolved asinc and corresponding Lagrangian interpolation at LC points.

t-value for the LC	Convolved asinc slope	Lagrangian interpolation slope
0.5615	–0.42526	–0.34025
2.1164	0.82206	0.82193
2.4505	0.95422	0.95417
2.76	0.966407	0.966401
3.084	**0.86716**	**0.86751**
3.4822	**0.62164**	**0.62082**

4.1.4 Slope Plots

Figure 3 gives the slopes of the original and Lagrangian interpolation results. They differ a little in the beginning but soon become indistinguishable.

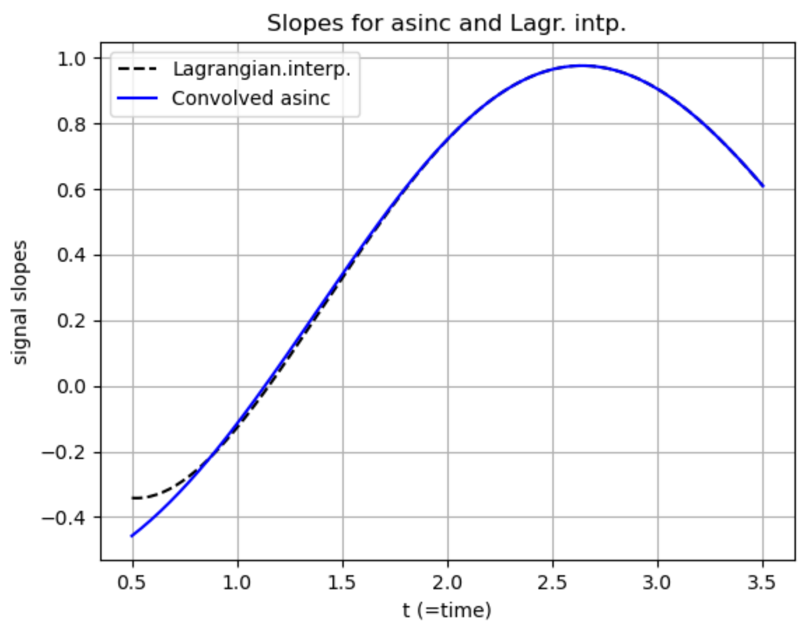

Figure 3. Slopes of the convolved asinc and Lagrangian interpolation functions for the first six points.

4.2 Last Six LC Intersection Points

4.2.1 Lagrangian Interpolation Input

Time values are obtained by inverse calculation from the convolved asinc function given above. They are as follows:

(time, LC): (0.5615, 0.9), (2.1164, 1.2), (2.4505, 1.5), (2.76, 1.8), (3.084, 2.1) (3.4822, 2.4)

4.2.2 Function Plots

Figure 4 shows the Lagrangian interpolation results for the last six points. The Lagrangian curve reproduces the original signal quite faithfully.

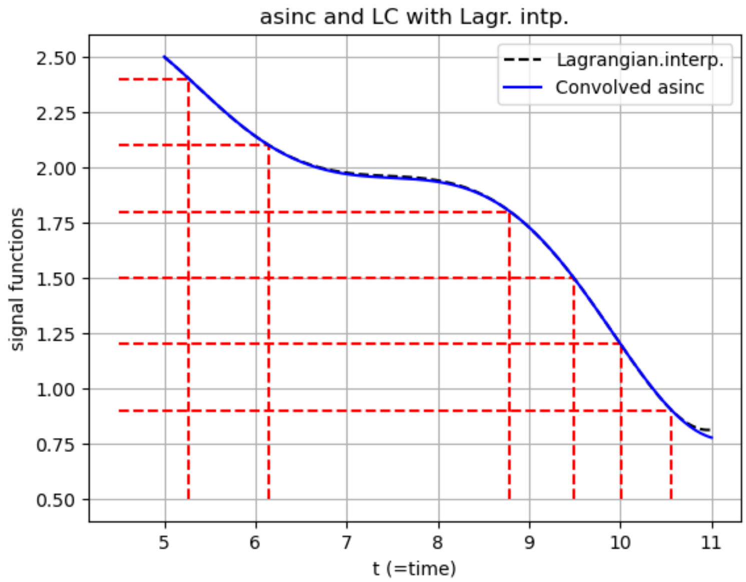

Figure 4. Convolved asinc function and its Lagrangian interpolation for the last six points.

4.2.3 Slope Values

Table 2 shows the original and calculated slope values for the last six points.

Table 2. Slopes of the convolved asinc and corresponding Lagrangian interpolation at the last six LC points.

t-value for the LC	Convolved asinc slope	Lagrangian interpolation slope
5.2731	–0.38237	–0.38365
6.1469	–0.25922	–0.25464
8.7906	–0.30067	–0.30579
9.4894	–0.54391	–0.54096
10.002	–1.19843	–0.60274
10.5635	–0.42526	–0.40509

4.2.4 Slope Plots

Figure 5 gives the slopes of the original and Lagrangian interpolation results for the last six points. They follow one another quite faithfully.

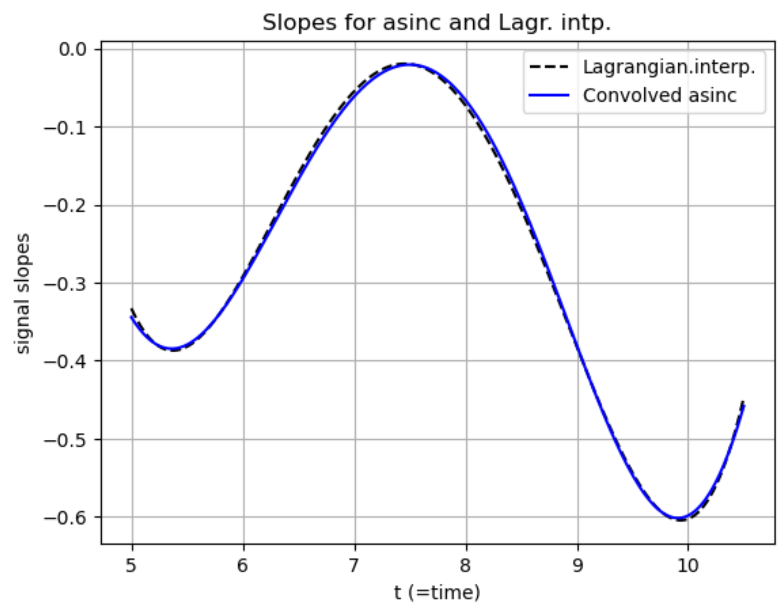

Figure 5. Slopes of the convolved asinc and Lagrangian interpolation functions for the last six points.

4.3 Discussion

The Lagrangian interpolation gives very satisfactory fits to the chosen convolved asinc function. The slope values are almost the same as well. This gives confidence that this method is a good alternative to other methods, e.g., PAR.

5. Conclusions and Future Directions

In this work, the Lagrangian interpolation method is used to obtain a CT envelope for LC signals. The reconstructed signal compares favorably with other similar methods,[14] especially regarding slope values. In the future, this method will be used for progressive reconstruction of LC signals.

6. References

1. Tsividis Y. Continuous-time digital signal processing. Electron Lett. 2003;39(21):1551. https://doi.org/10.1049/el:20031015
2. Tsividis Y. Event-driven data acquisition and digital signal processing: a tutorial. IEEE Trans Circuits Syst II Express Briefs. 2010;57(8):577–581. https://doi.org/10.1109/TCSII.2010.2056012
3. Vezyrtzis C, Tsividis Y. Processing of signals using level-crossing sampling. 2009 IEEE International Symposium on Circuits and Systems; 2009 June; Taipei, Taiwan. p. 2293–2296. https://doi.org/10.1109/ISCAS.2009.5118257
4. Kurchuk M. Signal encoding and digital signal processing in continuous time [dissertation]. Columbia University; 2011. https://doi.org/10.7916/D85T3SFB
5. Zanoli S, Ansaloni G, Teijeiro T, Atiena D. Mathematical model of parameters relevance in adaptive level-crossing sampling for electrocardiogram signals. arXiv; 2025 Jan 18. arXiv:2501.10829. https://doi.org/10.48550/arXiv.2501.10829
6. Marisa T et al. Pseudo asynchronous level crossing ADC for ECG signal acquisition. IEEE Trans Biomed Circuits Syst. 2017;11(2):267–278. https://doi.org/10.1109/TBCAS.2016.2619858
7. Crowe WM, Jungwirth P. Introduction to the progressive asinc reconstruction (PAR) technique for accurate real-time reconstruction of level-crossing (LC) signals. DEVCOM Aviation & Missile Center (US); 2026 Jan. Report No.: FCDD-AMT-25-13.
8. Jungwirth P, Crowe WM. Continuous time digital signal processing and signal reconstruction. IEEE Annual Computing and Communication Workshop and Conference (CCWC); 2023 Mar 8–11; Las Vegas, NV. p. 1205–1211. https://doi.org/10.1109/CCWC57344.2023.10099213
9. Riazifar N, Stocks NG. Alias-free level crossing sampling. EUSIPCO 2024. 32nd European Signal Processing Conference; 2024 Aug 26–30; Lyon, France.
10. Riazifar N, Stocks NG. Efficient signal recovery from level crossing sampling method. Circuits Syst Signal Process. 2026 Apr 2. https://doi.org/10.1007/s00034-026-03574-7
11. Maymon S, Oppenheim AV. Sinc interpolation of nonuniform samples. IEEE Trans Signal Process. 2011;59(10):4745–4758. https://doi.org/10.1109/TSP.2011.2160054

12. Selva J. Functionally weighted Lagrange interpolation of band-limited signals from nonuniform signals. IEEE Trans Signal Process. 2009;57(1):168–181. https://doi.org/10.1109/TSP.2008.2007101

13. Jiang H, Chen X, Chen L. Sinh regularized nonuniform sampling series. arXiV; 2026 Feb 10. arXiv:2601.17685v2 [math.NA]. https://doi.org/10.48550/arXiv.2601.17685

14. Mishra VK. An analytical approach to level-crossing-based sampling. DEVCOM Army Research Laboratory (US); 2025 Aug. Report No.: ARL-TR-10197.

List of Symbols, Abbreviations, and Acronyms

asinc	Aliased Cardinal Sine
CT	Continuous Time
DAC	Digital-to-Analog Conversion
DSP	Digital Signal Processing
DT	Discrete Time
LC	Level Crossing
LCS	Level-Crossing Sampled
LI	Lagrangian Interpolation
PAR	Progressive Asinc Reconstruction
sinc	Cardinal Sine

1 (PDF)	DEFENSE TECHNICAL INFORMATION CTR DTIC OCA
1 (PDF)	DEVCOM ARL FCDD RLB CB TECH LIB